

NON-UNITARY MINIMAL MODELS, BAILEY'S LEMMA AND $N = 1, 2$ SUPERCONFORMAL ALGEBRAS

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ABSTRACT. Using the Bailey flow construction, we derive character identities for the $N = 1$ superconformal models $SM(p', 2p + p')$ and $SM(p', 3p' - 2p)$, and the $N = 2$ superconformal model with central charge $c = 3(1 - \frac{2p}{p'})$ from the nonunitary minimal models $M(p, p')$. A new Ramond sector character formula for representations of $N = 2$ superconformal algebras with central element $c = 3(1 - \frac{2p}{p'})$ is given.

1. INTRODUCTION

Bailey's lemma is a powerful method to prove q -series identities of the Rogers–Ramanujan-type [3]. One of the key features of Bailey's lemma is its iterative structure which was first observed by Andrews [2] (see also [22]). This iterative structure called the Bailey chain makes it possible to start with one seed identity and derive an infinite family of identities from it. The Bailey chain has been generalized to the Bailey lattice [1] which yields a whole tree of identities from a single seed.

The relevance of the Andrews–Bailey construction to physics was first revealed in the papers by Foda and Quano [14, 15] in which they derived identities for the Virasoro characters using Bailey's lemma. By the application of Bailey's lemma to polynomial versions of the character identity of one conformal field theory, one obtains character identities of another conformal field theory. This relation between the two conformal field theories is called Bailey flow. In [4] it was demonstrated that there is a Bailey flow from the minimal models $M(p - 1, p)$ to $N = 1$ and $N = 2$ superconformal models. More precisely, it was shown that there is a Bailey flow from $M(p - 1, p)$ to $M(p, p + 1)$, and from $M(p - 1, p)$ to the $N = 1$ superconformal model $SM(p, p + 2)$ and the unitary $N = 2$ superconformal model with central charge $c = 3(1 - \frac{2}{p})$. In the conclusions of [4] it was mentioned that this construction can also be carried out for the nonunitary minimal models $M(p, p')$ where p and p' are relatively prime. In this paper we consider the nonunitary case. We show that starting with character identities for the nonunitary minimal model $M(p, p')$ of [6, 26], characters of the $N = 1$ superconformal models $SM(p', 2p + p')$, $SM(p', 3p' - 2p)$ and of the $N = 2$ superconformal model with central element $c = 3(1 - \frac{2p}{p'})$ can be obtained via the Bailey flow. We also give a new Ramond sector character formula for a representation of the $N = 2$ superconformal model with central element $c = 3(1 - \frac{2p}{p'})$.

The character identities obtained from the Bailey flow construction are of Bose-Fermi type. The bosonic side is associated with the construction of singular vectors of the underlying conformal field theory. The fermionic side is usually manifestly positive and reflects the quasiparticle structure of the model.

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The paper is organized as follows. In section 2 we provide the necessary background about Bailey pairs and fermionic formulas of the $M(p, p')$ models. This section is added to make this paper self-contained. For details the reader should consult [4, 6, 7]. In section 3 the characters of the $N = 1$ supersymmetric models $SM(2p + p', p')$ and $SM(3p' - 2p, p')$ are derived using the Bailey flow. Explicit fermionic expressions for these characters are given. In section 4 the background regarding $N = 2$ superconformal models is stated and a new character for the Ramond sector is derived. Then it is demonstrated how to obtain the characters of the $N = 2$ superconformal model with central element $c = 3(1 - \frac{2p}{p'})$ via the Bailey flow along with the explicit fermionic expressions for these characters. In section 5 we conclude with some remarks.

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2. BAILEY'S LEMMA

In this section we summarize Bailey's original lemma [2, 3] and the Bose-Fermi identities for the $M(p, p')$ minimal models [5, 6, 16, 26].

2.1. Bilateral Bailey lemma. A pair (α_n, β_n) of sequences $\{\alpha_n\}_{n \geq 0}$ and $\{\beta_n\}_{n \geq 0}$ is called a **Bailey pair** with respect to a if

$$(2.1) \quad \beta_n = \sum_{j=0}^n \frac{\alpha_j}{(q)_{n-j}(aq)_{n+j}}$$

where

$$(a)_n := (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k),$$

$$(a)_{-n} := (a; q)_{-n} = \frac{1}{\prod_{k=1}^n (1 - aq^{-k})}.$$

Following [4], we are going to use an extended definition in this paper called the bilateral Bailey pair. A pair (α_n, β_n) of sequences $\{\alpha_n\}_{n \in \mathbb{Z}}$ and $\{\beta_n\}_{n \in \mathbb{Z}}$ is said to be a **bilateral Bailey pair** with respect to a if

$$(2.2) \quad \beta_n = \sum_{j=-\infty}^n \frac{\alpha_j}{(q)_{n-j}(aq)_{n+j}}.$$

Theorem 2.1 (Bilateral Bailey lemma [2, 3, 4]). *If (α_n, β_n) is a bilateral Bailey pair then*

$$(2.3) \quad \sum_{n=-\infty}^{\infty} (\rho_1)_n (\rho_2)_n (aq/\rho_1\rho_2)^n \beta_n$$

$$= \frac{(aq/\rho_1)_{\infty} (aq/\rho_2)_{\infty}}{(aq)_{\infty} (aq/\rho_1\rho_2)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(\rho_1)_n (\rho_2)_n (aq/\rho_1\rho_2)^n \alpha_n}{(aq/\rho_1)_n (aq/\rho_2)_n}.$$

This lemma has been used with various Bailey pairs and different specializations of the parameters ρ_1 and ρ_2 to prove many q -series identities (see for example [1, 4, 15, 25]). In this paper the bilateral Bailey lemma is used to derive character identities for $N = 1, 2$ superconformal algebras from nonunitary minimal models $M(p, p')$.

A useful way to obtain new Bailey pairs from old ones is the construction of dual Bailey pairs. If (α_n, β_n) is a bilateral Bailey pair with respect to a , the **dual Bailey pair** (A_n, B_n) is defined as

$$(2.4) \quad \begin{aligned} A_n(a, q) &= a^n q^{n^2} \alpha_n(a^{-1}, q^{-1}), \\ B_n(a, q) &= a^{-n} q^{-n^2-n} \beta_n(a^{-1}, q^{-1}). \end{aligned}$$

Then (A_n, B_n) satisfies (2.2) with respect to a .

2.2. Bailey pairs from the minimal models $M(p, p')$. As shown by Foda and Quano [15], the Bose-Fermi character identities [5, 6, 16, 26] for the minimal models $M(p, p')$ are of the form

$$(2.5) \quad B_{r(b),s}(L, b; q) = q^{-\mathcal{N}_{r(b),s}} F_{r(b),s}(L, b; q),$$

with $\mathcal{N}_{r(b),s}$ as given in [6] and

$$(2.6) \quad \begin{aligned} B_{r(b),s}(L, b; q) &= \sum_{j=-\infty}^{\infty} \left(q^{j(jpp'+r(b)p'-sp)} \left[\begin{matrix} L \\ \frac{1}{2}(L+s-b)-jp' \end{matrix} \right]_q \right. \\ &\quad \left. - q^{(jp-r)(jp'-s)} \left[\begin{matrix} L \\ \frac{1}{2}(L-s-b)+jp' \end{matrix} \right]_q \right). \end{aligned}$$

Here

$$(2.7) \quad \left[\begin{matrix} n \\ j \end{matrix} \right]_q = \frac{(q)_n}{(q)_j (q)_{n-j}}$$

is the q -binomial coefficient. The function fermionic formula $F_{r(b),s}(L, b; q)$ will be discussed in the next section. For simplicity we are going to write r for $r(b)$. Following [15, 4] the identity (2.5) yields the bilateral Bailey pair relative to $a = q^{b-s+2x}$ where $x = \frac{L-2n-b+s}{2}$

$$(2.8) \quad \begin{aligned} \alpha_n &= \begin{cases} q^{j(jpp'+rp'-sp)} & \text{if } n = jp' - x \\ -q^{(jp-r)(jp'-s)} & \text{if } n = jp' - b - x \\ 0 & \text{otherwise} \end{cases} \\ \beta_n &= \frac{q^{-\mathcal{N}_{r,s}}}{(aq)_{2n}} F_{r,s}^{(p,p')}(2n+b-s+2x, b; q). \end{aligned}$$

The dual Bailey pair to (2.8) relative to $a = q^{b-s+2x}$ is

$$(2.9) \quad \begin{aligned} \hat{\alpha}_n &= \begin{cases} q^{j^2 p'(p'-p)-jp'(r-b)-js(p'-p)-x(b+x-s)} & \text{if } n = jp' - x \\ -q^{(jp'-s)(j(p'-p)+r-b)-x(b+x-s)} & \text{if } n = jp' - b - x \\ 0 & \text{otherwise} \end{cases} \\ \hat{\beta}_n &= \frac{q^{\mathcal{N}_{r,s}}}{(aq)_{2n}} a^n q^{n^2} F_{r,s}^{(p,p')}(2n+b-s+2x, b; q^{-1}). \end{aligned}$$

Inserting (2.8) and (2.9) into the bilateral Bailey lemma yields

$$\begin{aligned}
(2.10) \quad & \sum_{n=0}^{\infty} (\rho_1)_n (\rho_2)_n (aq/\rho_1\rho_2)^n \frac{q^{-\mathcal{N}_{r(b),s}}}{(aq)_{2n}} F_{r,s}^{(p,p')}(2n+b-s+2x,b;q) \\
&= \frac{(aq/\rho_1)_{\infty} (aq/\rho_2)_{\infty}}{(aq)_{\infty} (aq/\rho_1\rho_2)_{\infty}} \sum_{j=-\infty}^{\infty} \left(\frac{(\rho_1)_{jp'-x} (\rho_2)_{jp'-x}}{(aq/\rho_1)_{jp'-x} (aq/\rho_2)_{jp'-x}} (aq/\rho_1\rho_2)^{jp'-x} \right. \\
&\quad \times q^{j(jp'+rp'-sp)} - \frac{(\rho_1)_{jp'-b-x} (\rho_2)_{jp'-b-x}}{(aq/\rho_1)_{jp'-b-x} (aq/\rho_2)_{jp'-b-x}} \\
&\quad \left. \times (aq/\rho_1\rho_2)^{jp'-b-x} q^{(jp-r)(jp'-s)} \right)
\end{aligned}$$

and

$$\begin{aligned}
(2.11) \quad & \sum_{n=0}^{\infty} (\rho_1)_n (\rho_2)_n (aq/\rho_1\rho_2)^n \frac{q^{\mathcal{N}_{r(b),s}}}{(aq)_{2n}} a^n q^{n^2} F_{r,s}^{(p,p')}(2n+b-s+2x,b;q^{-1}) \\
&= \frac{(aq/\rho_1)_{\infty} (aq/\rho_2)_{\infty}}{(aq)_{\infty} (aq/\rho_1\rho_2)_{\infty}} \sum_{j=-\infty}^{\infty} \left(\frac{(\rho_1)_{jp'-x} (\rho_2)_{jp'-x}}{(aq/\rho_1)_{jp'-x} (aq/\rho_2)_{jp'-x}} (aq/\rho_1\rho_2)^{jp'-x} \right. \\
&\quad \times q^{j^2 p'(p'-p) - jp'(r-b) - js(p'-p) - x(b+x-s)} - \frac{(\rho_1)_{jp'-b-x} (\rho_2)_{jp'-b-x}}{(aq/\rho_1)_{jp'-b-x} (aq/\rho_2)_{jp'-b-x}} \\
&\quad \left. \times (aq/\rho_1\rho_2)^{jp'-b-x} q^{(jp'-s)(j(p'-p)+r-b) - x(b+x-s)} \right).
\end{aligned}$$

As in [4], we are going to consider different specializations of the parameters ρ_1 and ρ_2 in (2.10) and (2.11) to get character identities for $N = 1, 2$ superconformal algebras.

2.3. Fermionic formulas for $M(p, p')$. So far we have only considered the bosonic side of (2.5) explicitly. It suffices for the purpose of this paper to state the fermionic formula for the case $p < p' < 2p$ with p and p' relatively prime and r, s being pure Takahashi length. We follow [7, Section 4]. The fermionic formula depends on the continued fraction decomposition

$$\frac{p'}{p'-p} = 1 + \nu_0 + \frac{1}{\nu_1 + \frac{1}{\nu_2 + \cdots \frac{1}{\nu_{n_0} + 2}}}.$$

Define $t_i = \sum_{j=0}^{i-1} \nu_j$ for $1 \leq i \leq n_0 + 1$ and the fractional level incidence matrix \mathcal{I}_B and corresponding Cartan matrix B as

$$(\mathcal{I}_B)_{j,k} = \begin{cases} \delta_{j,k+1} + \delta_{j,k-1} & \text{for } 1 \leq j < t_{n_0+1}, j \neq t_i \\ \delta_{j,k+1} + \delta_{j,k} - \delta_{j,k-1} & \text{for } j = t_i, 1 \leq i \leq n_0 - \delta_{\nu_{n_0},0} \\ \delta_{j,k+1} + \delta_{\nu_{n_0},0} \delta_{j,k} & \text{for } j = t_{n_0+1} \end{cases}$$

$$B = 2I_{t_{n_0+1}} - \mathcal{I}_B,$$

where I_n is the identity matrix of dimension n . Recursively define

$$\begin{aligned}
y_{m+1} &= y_{m-1} + (\nu_m + \delta_{m,0} + 2\delta_{m,n_0})y_m, & y_{-1} &= 0, & y_0 &= 1, \\
\overline{y}_{m+1} &= \overline{y}_{m-1} + (\nu_m + \delta_{m,0} + 2\delta_{m,n_0})\overline{y}_m, & \overline{y}_{-1} &= -1, & \overline{y}_0 &= 1.
\end{aligned}$$

Then the Takahashi length and truncated Takahashi length are given by

$$\frac{\ell_{j+1}}{\bar{\ell}_{j+1}} = \frac{y_{m-1} + (j - t_m)y_m}{\bar{y}_{m-1} + (j - t_m)\bar{y}_m} \quad \text{for } t_m < j \leq t_{m+1} + \delta_{m,n_0} \text{ with } 0 \leq m \leq n_0.$$

For $b = \ell_{\beta+1}$, $r(b) = \bar{\ell}_{\beta+1}$ with $t_\xi < \beta \leq t_{\xi+1} + \delta_{\xi,n_0}$ and $s = \ell_{\sigma+1}$ with $t_\zeta < \sigma \leq t_{\zeta+1} + \delta_{\zeta,n_0}$ the fermionic formula is given by

$$(2.12) \quad F_{r,s}^{(p,p')}(L, b; q) = q^{k_{b,s}} \sum_{\mathbf{m} \equiv Q_{\mathbf{u},\mathbf{v}} \pmod{2}} q^{\frac{1}{4}\mathbf{m}^t B \mathbf{m} - \frac{1}{2}A_{\mathbf{u},\mathbf{v}} \mathbf{m}} \prod_{j=1}^{t_{n_0+1}} \left[\begin{matrix} n_j + m_j \\ m_j \end{matrix} \right]_q'$$

where $k_{b,s}$ is a normalization constant and $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^{t_{n_0+1}}$ such that

$$(2.13) \quad \mathbf{n} + \mathbf{m} = \frac{1}{2}(\mathcal{I}_B \mathbf{m} + \mathbf{u} + \mathbf{v} + L\mathbf{e}_1)$$

with \mathbf{e}_i the standard i -th basis element of $\mathbb{Z}^{t_{n_0+1}}$, $\mathbf{u} = \mathbf{e}_\beta - \sum_{k=\xi+1}^{n_0} \mathbf{e}_{t_k}$, $\mathbf{v} = \mathbf{e}_\sigma - \sum_{k=\zeta+1}^{n_0} \mathbf{e}_{t_k}$ and $Q_{\mathbf{u},\mathbf{v}}$, $A_{\mathbf{u},\mathbf{v}}$ as defined in [7, Section 4.2]. The q -binomial is also defined for negative entries

$$\left[\begin{matrix} n + m \\ m \end{matrix} \right]_q' = \frac{(q^{n+1})_m}{(q)_m}.$$

Note that

$$(2.14) \quad \left[\begin{matrix} n + m \\ m \end{matrix} \right]_{q^{-1}}' = q^{-nm} \left[\begin{matrix} n + m \\ m \end{matrix} \right]_q'.$$

In fact using (2.14) we get the following dual form of the fermionic formula that will be useful later on

$$(2.15) \quad F_{r,s}^{(p,p')}(L, b; q^{-1}) = q^{-k_{b,s}} \sum_{\mathbf{m} \equiv Q_{\mathbf{u},\mathbf{v}}} q^{\frac{1}{4}\mathbf{m}^t B \mathbf{m} - \frac{1}{2}L m_1 + \frac{1}{2}A_{\mathbf{u},\mathbf{v}} \mathbf{m} - \frac{1}{2}\mathbf{m}^t(\mathbf{u} + \mathbf{v})} \prod_{j=1}^{t_{n_0+1}} \left[\begin{matrix} n_j + m_j \\ m_j \end{matrix} \right]_q'.$$

3. $N = 1$ SUPERCONFORMAL CHARACTER FROM $M(p, p')$

In this section we are going to consider the specialization in (2.10) and (2.11)

$$(3.1) \quad \rho_1 \longrightarrow \infty, \quad \rho_2 = \text{finite}.$$

We will see that these give characters of the $N = 1$ superconformal model $SM(p, p')$ given by [10, 17],

$$(3.2) \quad \tilde{\chi}_{r,s}^{(p,p')}(q) = \tilde{\chi}_{p-r,p'-s}^{(p,p')}(q) = \frac{(-q^{\epsilon_{r-s}})_\infty}{(q)_\infty} \sum_{j=-\infty}^{\infty} \left(q^{\frac{j(jpp' + rp' - sp)}{2}} - q^{\frac{(jp-r)(jp'-s)}{2}} \right),$$

where $1 \leq r \leq p-1$, $1 \leq s \leq p'-1$, p and $(p'-p)/2$ are relatively prime and

$$(3.3) \quad \epsilon_i = \begin{cases} \frac{1}{2} & \text{if } i \text{ is even (NS-sector),} \\ 1 & \text{if } i \text{ is odd (R-sector).} \end{cases}$$

The central charge is $c = \frac{3}{2} - \frac{3(p-p')^2}{pp'}$.

3.1. The model $SM(p', 2p + p')$. Specializing $\rho_1 \rightarrow \infty$ and $\rho_2 = -q^{\frac{b-s+1}{2}}$ with $x = 0$ in (2.10) we find for $b - s$ even (NS sector)

$$(3.4) \quad \tilde{\chi}_{s,2r+b}^{(p', 2p+p')}(q) = \sum_{n \geq 0} \frac{q^{\frac{1}{2}(n^2 + nb - ns)} (-q^{\frac{1}{2}})_{n+(b-s)/2}}{(q)_{2n+b-s}} q^{-\mathcal{N}_{r,s}} F_{r,s}^{(p,p')}(2n+b-s, b; q)$$

and for $b - s$ odd (R-sector)

$$(3.5) \quad \tilde{\chi}_{s,2r+b}^{(p', 2p+p')}(q) = \sum_{n \geq 0} \frac{q^{\frac{1}{2}(n^2 + nb - ns)} (-q)_{n+(b-s-1)/2}}{(q)_{2n+b-s}} q^{-\mathcal{N}_{r,s}} F_{r,s}^{(p,p')}(2n+b-s, b; q).$$

To obtain an explicit fermionic formula set $m_0 = L = 2n + b - s$ and insert (2.12) into (3.4). Then using

$$(3.6) \quad (-q^{\frac{1}{2}})_{\frac{m_0}{2}} = \sum_{k=0}^{\frac{m_0}{2}} q^{\frac{1}{2}(\frac{m_0}{2}-k)^2} \left[\frac{\frac{m_0}{2}}{k} \right]_q$$

we find

$$(3.7) \quad \begin{aligned} \tilde{\chi}_{s,2r+b}^{(p', 2p+p')}(q) &= q^{-\frac{1}{8}(b-s)^2 - \mathcal{N}_{r,s} + k_{b,s}} \sum_{\substack{m_0=0 \\ m_0 \text{ even}}}^{\infty} \sum_{k=0}^{\frac{m_0}{2}} \sum_{\mathbf{m} \equiv \bar{Q}_{\mathbf{u},\mathbf{v}}} q^{\frac{1}{8}m_0^2 + \frac{1}{2}(\frac{m_0}{2}-k)^2} \\ &\times q^{\frac{1}{4}\mathbf{m}^t B \mathbf{m} - \frac{1}{2}A_{\mathbf{u},\mathbf{v}} \mathbf{m}} \times \frac{1}{(q)_{m_0}} \left[\frac{\frac{m_0}{2}}{k} \right]_q \prod_{j=1}^{t_{n_0+1}} \left[\begin{matrix} n_j + m_j \\ m_j \end{matrix} \right]_q'. \end{aligned}$$

Setting $\mathbf{p} = (k, m_0, \mathbf{m}) \in \mathbb{Z}^{t_{n_0+1}+2}$, (3.7) in the NS-sector can be rewritten as

$$(3.8) \quad \begin{aligned} \tilde{\chi}_{s,2r+b}^{(p', 2p+p')}(q) &= q^{-\frac{1}{8}(b-s)^2 - \mathcal{N}_{r,s} + k_{b,s}} \sum_{\substack{\mathbf{p} \in \mathbb{Z}^{t_{n_0+1}+2} \\ p_i \equiv (\bar{Q}_{\mathbf{u},\mathbf{v}})_i, i \geq 2}} q^{\frac{1}{4}\mathbf{p}^t \tilde{B} \mathbf{p} - \frac{1}{2}\tilde{A}_{\mathbf{u},\mathbf{v}} \mathbf{p}} \\ &\times \frac{1}{(q)_{p_2}} \prod_{j=1, j \neq 2}^{t_{n_0+1}+2} \left[\begin{matrix} \frac{1}{2}(\mathcal{I}_{\tilde{B}} \mathbf{p} + \tilde{\mathbf{u}} + \tilde{\mathbf{v}})_j \\ p_j \end{matrix} \right]_q' \end{aligned}$$

where $\mathcal{I}_{\tilde{B}} = 2I_{t_{n_0+1}+2} - \tilde{B}$,

$$(3.9) \quad \begin{aligned} \tilde{B} &= \left(\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & B \end{array} \right) \\ \tilde{A}_{\mathbf{u},\mathbf{v}} &= (0, 0, A_{\mathbf{u},\mathbf{v}}), \\ \tilde{\mathbf{u}}^t &= (0, 0, \mathbf{u}^t), \\ \tilde{\mathbf{v}}^t &= (0, 0, \mathbf{v}^t), \\ \tilde{Q}_{\mathbf{u},\mathbf{v}}^t &= (0, 0, Q_{\mathbf{u},\mathbf{v}}^t). \end{aligned}$$

Similarly setting $m_0 = 2n + b - s$ in (3.5) and using

$$(3.10) \quad (-q)_{\frac{m_0-1}{2}} = \frac{1}{2} \sum_{k=0}^{\frac{m_0+1}{2}} q^{\frac{1}{2}(\frac{m_0+1}{2}-k)(\frac{m_0-1}{2}-k)} \left[\frac{\frac{m_0+1}{2}}{k} \right]_q$$

we get the fermionic formula in the R-sector,

$$(3.11) \quad \tilde{\chi}_{s,2r+b}^{(p',2p+p')}(q) = \frac{1}{2} q^{-\frac{1}{8}((b-s)^2+1)-\mathcal{N}_{r,s}+k_{b,s}} \sum_{\substack{\mathbf{p} \in \mathbb{Z}^{t_{n_0+1}+2} \\ p_i \equiv (\tilde{Q}_{\mathbf{u},\mathbf{v}})_i, i \geq 2}} q^{\frac{1}{4}\mathbf{p}^t \tilde{B} \mathbf{p} - \frac{1}{2}\tilde{A}_{\mathbf{u},\mathbf{v}} \mathbf{p}} \\ \times \frac{1}{(q)_{p_2}} \prod_{j=1, j \neq 2}^{t_{n_0+1}+2} \left[\frac{\frac{1}{2}(\mathcal{I}_{\tilde{B}} \mathbf{p} + \tilde{\mathbf{u}} + \tilde{\mathbf{v}})_j}{p_j} \right]_q'$$

where $\tilde{B}, \tilde{A}, \tilde{\mathbf{v}}$ are as in (3.9) and $\tilde{\mathbf{u}}^t = (1, 0, \mathbf{u}^t)$, $\tilde{Q}_{\mathbf{u},\mathbf{v}}^t = (0, 1, Q_{\mathbf{u},\mathbf{v}}^t)$.

3.2. The model $SM(p', 3p' - 2p)$. Similarly using the same specialization with the dual Bailey pair in (2.11) we find for $b - s$ even in the NS-sector

$$(3.12) \quad \tilde{\chi}_{s,3b-2r}^{(p',3p'-2p)}(q) = \sum_{n \geq 0} \frac{q^{\frac{3n}{2}(n+b-s)} (-q^{\frac{1}{2}})_{n+(b-s)/2}}{(q)_{2n+b-s}} q^{\mathcal{N}_{r,s}} F_{r,s}^{(p,p')}(2n+b-s, b; q^{-1})$$

and for $b - s$ odd in the R-sector

$$(3.13) \quad \tilde{\chi}_{s,3b-2r}^{(p',3p'-2p)}(q) = \sum_{n \geq 0} \frac{q^{\frac{3n}{2}(n+b-s)} (-q)_{n+(b-s-1)/2}}{(q)_{2n+b-s}} q^{\mathcal{N}_{r,s}} F_{r,s}^{(p,p')}(2n+b-s, b; q^{-1}).$$

To obtain the fermionic formula, as before we are going to set $m_0 = 2n + b - s$. Inserting (3.10) and (2.15) into (3.13) we get in the R-sector

$$(3.14) \quad \tilde{\chi}_{s,3b-2r}^{(p',3p'-2p)}(q) = \frac{1}{2} q^{-\frac{1}{8}(3(b-s)^2+1)+\mathcal{N}_{r,s}-k_{b,s}} \sum_{\substack{m_0=0 \\ m_0 \text{ odd}}}^{\infty} \sum_{k=0}^{\frac{m_0+1}{2}} \sum_{\mathbf{m} \equiv Q_{\mathbf{u},\mathbf{v}}} q^{\frac{1}{2}(m_0^2+k^2-m_0k-m_0m_1)} q^{\frac{1}{4}\mathbf{m}^t B \mathbf{m} - \frac{1}{2}\mathbf{m}^t(\mathbf{u}+\mathbf{v}) + \frac{1}{2}A_{\mathbf{u},\mathbf{v}} \mathbf{m}} \\ \times \frac{1}{(q)_{m_0}} \left[\frac{\frac{m_0+1}{2}}{k} \right]_q \prod_{j=1}^{t_{n_0+1}} \left[\frac{n_j + m_j}{m_j} \right]_q'.$$

Define $\mathbf{p} = (k, m_0, \mathbf{m}) \in \mathbb{Z}^{t_{n_0+1}+2}$, so that (3.14) in the R-sector can be rewritten as

$$(3.15) \quad \tilde{\chi}_{s,3b-2r}^{(p',3p'-2p)}(q) = \frac{1}{2} q^{-\frac{1}{8}(3(b-s)^2+1)+\mathcal{N}_{r,s}-k_{b,s}} \sum_{\substack{\mathbf{p} \in \mathbb{Z}^{t_{n_0+1}+2} \\ p_i \equiv (\tilde{Q}'_{\mathbf{u},\mathbf{v}})_i, i \geq 2}} q^{\frac{1}{4}\mathbf{p}^t \tilde{B}' \mathbf{p} + \frac{1}{2}\tilde{A}_{\mathbf{u},\mathbf{v}} \mathbf{p}} \\ \times \frac{1}{(q)_{p_2}} \prod_{j=1, j \neq 2}^{t_{n_0+1}+2} \left[\frac{\frac{1}{2}(\mathcal{I}_{\tilde{B}'} \mathbf{p} + \tilde{\mathbf{u}} + \tilde{\mathbf{v}})_j}{p_j} \right]_q'$$

where $\mathcal{I}_{\tilde{B}'} = 2I_{t_{n_0+1}+2} - \tilde{B}'$, $\tilde{\mathbf{v}}$ as in (3.9), $\tilde{\mathbf{u}}^t = (1, 0, \mathbf{u}^t)$, $(\tilde{Q}'_{\mathbf{u},\mathbf{v}})^t = (0, 1, Q_{\mathbf{u},\mathbf{v}}^t)$, and

$$(3.16) \quad \tilde{B}' = \left(\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & B \end{array} \right) \\ \tilde{A}_{\mathbf{u},\mathbf{v}} = (0, 0, A_{\mathbf{u},\mathbf{v}} - \mathbf{u}^t - \mathbf{v}^t).$$

Similarly, for the NS-sector it follows from (3.12)

$$(3.17) \quad \tilde{\chi}_{s,3b-2r}^{(p',3p'-2p)}(q) = q^{-\frac{3}{8}(b-s)^2 + \mathcal{N}_{r,s} - k_{b,s}} \sum_{\substack{\mathbf{p} \in \mathbb{Z}^{t_{n_0+1}+2} \\ p_i \equiv (\tilde{Q}'_{\mathbf{u},\mathbf{v}})_i, i \geq 2}} q^{\frac{1}{4}\mathbf{p}^t \tilde{B}' \mathbf{p} + \frac{1}{2}\tilde{A}_{\mathbf{u},\mathbf{v}} \mathbf{p}} \\ \times \frac{1}{(q)_{p_2}} \prod_{j=1, j \neq 2}^{t_{n_0+1}+2} \left[\frac{\frac{1}{2}(\mathcal{I}_{\tilde{B}'} \mathbf{p} + \tilde{\mathbf{u}} + \tilde{\mathbf{v}})_j}{p_j} \right]_q'$$

with \tilde{B}' and $\tilde{A}_{\mathbf{u},\mathbf{v}}$ as in (3.16), $(\tilde{Q}'_{\mathbf{u},\mathbf{v}})^t = (0, 0, Q_{\mathbf{u},\mathbf{v}}^t)$, $\tilde{\mathbf{u}}^t = (0, 0, \mathbf{u}^t)$ and $\tilde{\mathbf{v}}^t = (0, 0, \mathbf{v}^t)$.

4. $N = 2$ CHARACTER FORMULAS

4.1. $N = 2$ **superconformal algebra and Spectral flow.** The $N = 2$ superconformal algebra \mathcal{A} is the infinite dimensional Lie super algebra [13] with basis L_n, T_n, G_r^\pm, C and (anti)-commutation relation given by

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0} \\ [L_m, G_r^\pm] &= \left(\frac{1}{2}m - r\right)G_{m+r}^\pm \\ [L_m, T_n] &= -nT_{m+n} \\ [T_m, T_n] &= \frac{1}{3}cm\delta_{m+n,0} \\ [T_m, G_r^\pm] &= \pm G_{m+r}^\pm \\ \{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)T_{r+s} + \frac{C}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \\ [L_m, C] &= [T_n, C] = [G_r^\pm, C] = 0 \\ \{G_r^+, G_s^+\} &= \{G_r^-, G_s^-\} = 0 \end{aligned}$$

where $n, m \in \mathbb{Z}$, but r, s are integers in R-sector and half-integer in NS-sector. The element C is the central element and its eigenvalue c is parametrized as $c = 3(1 - \frac{2p}{p'})$, where p, p' are relatively prime positive integers.

It was observed in [18, 24] that there exists a family of outer automorphisms $\alpha_\eta : \mathcal{A} \rightarrow \mathcal{A}$ which maps the $N = 2$ superconformal algebras to itself. These are explicitly given by

$$(4.1) \quad \begin{aligned} \alpha_\eta(G_r^+) &= \hat{G}_r^+ = G_{r-\eta}^+ \\ \alpha_\eta(G_r^-) &= \hat{G}_r^- = G_{r+\eta}^- \\ \alpha_\eta(L_n) &= \hat{L}_n = L_n - \eta T_n + \frac{c}{6}\eta^2\delta_{n,0} \\ \alpha_\eta(T_n) &= \hat{T}_n = T_n - \frac{c}{3}\eta\delta_{n,0} \end{aligned}$$

This family of automorphisms is called **spectral flow** and $\eta \in \mathbb{R}$ is called the **flow parameter**. When $\eta \in \mathbb{Z}$ each sector of the algebra is mapped to itself. When $\eta \in \mathbb{Z} + \frac{1}{2}$ the Neveu-Schwarz sector is mapped to the Ramond sector and vice-versa. We are going to use the spectral flow $\eta = \pm \frac{1}{2}$ to map the NS-sector to the R-sector.

4.2. Spectral flow and characters. We denote the Verma module generated from a highest weight state $|h, Q, c\rangle$ with L_0 eigenvalue h , T_0 eigenvalue Q and central charge c by $V_{h,Q}$. The character $\chi_{V_{h,Q}}$ of a highest weight representation $V_{h,Q}$ is defined as

$$\chi_{V_{h,Q}}(q, z) = \text{Tr}_{V_{h,Q}}(q^{L_0 - c/24} z^{T_0}).$$

Following [18] the character transforms under the spectral flow in the following way

$$(4.2) \quad \text{Tr}_{V_{h,Q}}(q^{\hat{L}_0 - c/24} z^{\hat{T}_0}) = \text{Tr}_{V_{h^\eta, Q^\eta}}(q^{L_0 - c/24} z^{T_0}),$$

where h^η and Q^η are the eigenvalues of \hat{L}_0 and \hat{T}_0 , respectively, as defined in (4.1). This means the new character $\chi_{V_{h^\eta, Q^\eta}}(q, z)$ which is the trace of the transformed operators over the original representation equals the character of the representation defined by the eigenvalues h^η and Q^η of \hat{L}_0 and \hat{T}_0 , respectively. So the new character is the character of the representation V_{h^η, Q^η} .

For $\eta = \frac{1}{2}$ the spectral flow $\alpha_{\frac{1}{2}}$ takes a NS-sector character to an R-sector character. Let $\chi_{V_{h,Q}}^{NS}(q, z)$ be a NS-sector character corresponding to the representation $V_{h,Q}$. Then by (4.2) and (4.1) the new R-sector character $\chi_{V_{h^\eta, Q^\eta}}^R(q, z)$ is derived using

$$(4.3) \quad \begin{aligned} \chi_{V_{h^\eta, Q^\eta}}^R(q, z) &= \text{Tr}_{V_{h,Q}}(q^{\hat{L}_0 - c/24} z^{\hat{T}_0}) = \text{Tr}_{V_{h,Q}}(q^{L_0 - \frac{1}{2}T_0 + \frac{c}{24} - \frac{c}{24}} z^{T_0 - \frac{c}{6}}) \\ &= q^{\frac{c}{24}} z^{-\frac{c}{6}} \text{Tr}_{V_{h,Q}}(q^{L_0 - \frac{c}{24}} (zq^{-\frac{1}{2}})^{T_0}) = q^{\frac{c}{24}} z^{-\frac{c}{6}} \chi_{V_{h,Q}}^{NS}(q, zq^{-\frac{1}{2}}). \end{aligned}$$

4.3. R-sector character from NS-sector character. To simplify notation we are going to use a slightly different notation for characters. Since we are only dealing with the vacuum character in the NS-sector for which $h = 0, Q = 0$, we write $\hat{\chi}_{p,p'}^{NS}(q, z)$. The R-sector character is denoted by $\hat{\chi}_{p,p'}^R(q, z)$ with the corresponding (h, Q) specified separately.

Following [9, 12, 13, 18, 19] the vacuum character for the $N = 2$ superconformal algebra with central element $c = 3(1 - \frac{2p}{p'})$ in the NS-sector is given by

$$(4.4) \quad \begin{aligned} \hat{\chi}_{p,p'}^{NS}(q, z) &= q^{-c/24} \prod_{n=1}^{\infty} \frac{(1 + zq^{n-\frac{1}{2}})(1 + z^{-1}q^{n-\frac{1}{2}})}{(1 - q^n)^2} \\ &\times \left(1 - \sum_{n=0}^{\infty} \left(q^{p(n+1)(p'(n+1)-1)} + \frac{zq^{p'n(pn+1)+pn+\frac{1}{2}}}{1 + zq^{p'n+\frac{1}{2}}} + \frac{z^{-1}q^{p'n(pn+1)+pn+\frac{1}{2}}}{1 + z^{-1}q^{p'n+\frac{1}{2}}} \right) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \left(q^{pn(p'n+1)} + \frac{zq^{p'n(pn+1)-pn-\frac{1}{2}}}{1 + zq^{p'n-\frac{1}{2}}} + \frac{z^{-1}q^{p'n(pn+1)-pn-\frac{1}{2}}}{1 + z^{-1}q^{p'n-\frac{1}{2}}} \right) \right). \end{aligned}$$

This formula can be verified using the embedding diagram for the vacuum character as described in [13, 18] and can be rewritten as (as will be useful later)

$$(4.5) \quad \begin{aligned} \hat{\chi}_{p,p'}^{NS}(q, z) &= q^{-c/24} \prod_{n=1}^{\infty} \frac{(1 + zq^{n-\frac{1}{2}})(1 + z^{-1}q^{n-\frac{1}{2}})}{(1 - q^n)^2} \\ &\times \sum_{j=-\infty}^{\infty} q^{pj(p'j+1)} \frac{1 - q^{2p'j+1}}{(1 + zq^{p'j+\frac{1}{2}})(1 + z^{-1}q^{p'j+\frac{1}{2}})}. \end{aligned}$$

The unitary case $p = 1$ of these character formulas was given in [11, 20, 21, 23]. In particular if we put $z = 1$ in (4.5) we obtain the following formula derived in [13]

$$(4.6) \quad \hat{\chi}_{p,p'}^{NS}(q) = q^{-c/24} \prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{1}{2}})^2}{(1 - q^n)^2} \sum_{j=-\infty}^{\infty} q^{pj(p'j+1)} \frac{1 - q^{p'j+\frac{1}{2}}}{1 + q^{p'j+\frac{1}{2}}}.$$

Let us apply (4.3) to the NS-sector vacuum character (4.5) to get a Ramond sector character. From (4.1) it follows that

$$\begin{aligned}\hat{L}_0 &= L_0 - \frac{1}{2}T_0 + \frac{c}{24} \\ \hat{T}_0 &= T_0 - \frac{c}{6}.\end{aligned}$$

For the vacuum character in the NS-sector $(h, Q) = (0, 0)$, so the new eigenvalues are $(h^\eta, Q^\eta) = (\frac{c}{24}, -\frac{c}{6})$ in the R-sector. Hence the new character in the R-sector corresponds to (h^η, Q^η) and by (4.3)

$$\begin{aligned}(4.7) \quad \hat{\chi}_{p,p'}^R(q, z) &= q^{\frac{c}{24}} z^{-\frac{c}{6}} \hat{\chi}_{p,p'}^{NS}(q, zq^{-\frac{1}{2}}) \\ &= z^{-\frac{c}{6}} \frac{(-z)_\infty (-z^{-1}q)_\infty}{(q)_\infty^2} \sum_{j=-\infty}^{\infty} q^{pj(p'+1)} \frac{1 - q^{2p'+1}}{(1 + zq^{p'j})(1 + z^{-1}q^{p'j+1})}.\end{aligned}$$

4.4. $N = 2$ **superconformal characters for $c = 3(1 - \frac{2p}{p'})$.** Using $r = 0$ and $b = 1$ in (2.10) we obtain

$$\begin{aligned}(4.8) \quad &\sum_{n=0}^{\infty} (\rho_1)_n (\rho_2)_n (aq/\rho_1\rho_2)^n \frac{q^{-\mathcal{N}_{0,s}}}{(aq)_{2n}} F_{0,s}^{(p,p')}(2n+1-s+2x, 1; q) \\ &= \frac{(aq/\rho_1)_\infty (aq/\rho_2)_\infty}{(aq)_\infty (aq/\rho_1\rho_2)_\infty} \sum_{j=-\infty}^{\infty} \left(\frac{(\rho_1)_{jp'-x} (\rho_2)_{jp'-x}}{(aq/\rho_1)_{jp'-x} (aq/\rho_2)_{jp'-x}} (aq/\rho_1\rho_2)^{jp'-x} \right. \\ &\quad \left. - \frac{(\rho_1)_{jp'-1-x} (\rho_2)_{jp'-1-x}}{(aq/\rho_1)_{jp'-1-x} (aq/\rho_2)_{jp'-1-x}} (aq/\rho_1\rho_2)^{jp'-1-x} \right) q^{jp(jp'-s)}.\end{aligned}$$

In this section we consider the specialization

$$\rho_1 = \text{finite}, \quad \rho_2 = \text{finite}.$$

Taking the limit $\frac{aq}{\rho_1\rho_2} \rightarrow 1$ in (4.8), we find

$$\begin{aligned}(4.9) \quad &\sum_{n=0}^{\infty} (\rho_1)_n (\rho_2)_n \frac{q^{-\mathcal{N}_{0,s}}}{(aq)_{2n}} F_{0,s}^{(p,p')}(2n+1-s+2x, 1; q) \\ &= \frac{(\rho_1)_\infty (\rho_2)_\infty}{(\rho_1\rho_2)_\infty (q)_\infty} \sum_{j=-\infty}^{\infty} q^{jp(jp'-s)} \frac{\rho_1\rho_2 q^{2(jp'-x-1)} - 1}{(1 - \rho_1 q^{jp'-x-1})(1 - \rho_2 q^{jp'-x-1})}.\end{aligned}$$

4.5. **NS-sector characters.** Let us set $\rho_1 = -zq^{x+\frac{1}{2}}$, $\rho_2 = -z^{-1}q^{x+\frac{1}{2}}$ in (4.9), which implies $a = q^{2x}$ and $s = 1$. Making the variable change $j \rightarrow -j$ in (4.9) and setting $x = 0$ we obtain

$$\begin{aligned}(4.10) \quad &\sum_{n=0}^{\infty} (-zq^{\frac{1}{2}})_n (-z^{-1}q^{\frac{1}{2}})_n \frac{q^{-\mathcal{N}_{0,1}}}{(q)_{2n}} F_{0,1}^{(p,p')}(2n, 1; q) \\ &= \frac{(-zq^{\frac{1}{2}})_\infty (-z^{-1}q^{\frac{1}{2}})_\infty}{(q)_\infty^2} \sum_{j=-\infty}^{\infty} q^{jp(jp'+1)} \frac{1 - q^{2jp'+1}}{(1 + zq^{jp'+\frac{1}{2}})(1 + z^{-1}q^{jp'+\frac{1}{2}})}.\end{aligned}$$

Comparing with (4.5), we obtain

$$(4.11) \quad \hat{\chi}_{p,p'}^{NS}(q, z) = q^{-\frac{c}{24} - \mathcal{N}_{0,1}} \sum_{n=0}^{\infty} \frac{(-zq^{\frac{1}{2}})_n (-z^{-1}q^{\frac{1}{2}})_n}{(q)_{2n}} F_{0,1}^{(p,p')}(2n, 1; q).$$

Setting $z = 1$ and inserting the fermionic formula (2.12), we find

$$(4.12) \quad \hat{\chi}_{p,p'}^{NS}(q) = q^{-\frac{c}{24} - \mathcal{N}_{0,1} + k_{1,1}} \sum_{n=0}^{\infty} \left(\frac{(-q^{\frac{1}{2}})_n}{(q)_{2n}} \sum_{\mathbf{m} \equiv Q_{\mathbf{u},\mathbf{v}}} q^{\frac{1}{4} \mathbf{m}^t B \mathbf{m} - \frac{1}{2} A_{\mathbf{u},\mathbf{v}} \mathbf{m}} \times \prod_{j=1}^{t_{n_0}+1} \left[\begin{matrix} n_j + m_j \\ m_j \end{matrix} \right]_q' \right).$$

Let us set $m_0 = 2n$ and use (3.6) to get

$$(4.13) \quad \hat{\chi}_{p,p'}^{NS}(q) = q^{-\frac{c}{24} - \mathcal{N}_{0,1} + k_{1,1}} \sum_{\substack{m_0=0 \\ m_0 \text{ even}}}^{\infty} \sum_{k_1=0}^{\frac{m_0}{2}} \sum_{k_2=0}^{\frac{m_0}{2}} \sum_{\mathbf{m} \equiv Q_{\mathbf{u},\mathbf{v}}} q^{\frac{1}{2} (\frac{m_0}{2} - k_1)^2 + \frac{1}{2} (\frac{m_0}{2} - k_2)^2} \times q^{\frac{1}{4} \mathbf{m}^t B \mathbf{m} - \frac{1}{2} A_{\mathbf{u},\mathbf{v}} \mathbf{m}} \frac{1}{(q)_{m_0}} \left[\begin{matrix} \frac{m_0}{2} \\ k_1 \end{matrix} \right]_q \left[\begin{matrix} \frac{m_0}{2} \\ k_2 \end{matrix} \right]_q \prod_{j=1}^{t_{n_0}+1} \left[\begin{matrix} n_j + m_j \\ m_j \end{matrix} \right]_q'.$$

Define $\mathbf{p} = (k_1, k_2, m_0, \mathbf{m}) \in \mathbb{Z}^{t_{n_0}+1+3}$, so that (4.13) can be rewritten as

$$(4.14) \quad \hat{\chi}_{p,p'}^{NS}(q) = q^{-\frac{c}{24} - \mathcal{N}_{0,1} + k_{1,1}} \sum_{\substack{\mathbf{p} \in \mathbb{Z}^{t_{n_0}+1+3} \\ p_i \equiv (Q_{\mathbf{u},\mathbf{v}})_i, i \geq 3}} q^{\frac{1}{4} \mathbf{p}^t D \mathbf{p} - \frac{1}{2} \hat{A}_{\mathbf{u},\mathbf{v}} \mathbf{p}} \times \frac{1}{(q)_{p_3}} \prod_{j=1, j \neq 3}^{t_{n_0}+1+3} \left[\begin{matrix} \frac{1}{2} (\mathcal{I}_D \mathbf{p} + \hat{\mathbf{u}} + \hat{\mathbf{v}})_j \\ p_j \end{matrix} \right]_q',$$

where $\mathcal{I}_D = 2I_{t_{n_0}+1+3} - D$ and

$$(4.15) \quad \begin{aligned} D &= \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 1 & 1 \\ \hline 0 & 0 & -1 & B \end{array} \right), \\ \hat{A}_{\mathbf{u},\mathbf{v}} &= (0, 0, 0, A_{\mathbf{u},\mathbf{v}}), \\ \hat{\mathbf{u}}^t &= (0, 0, 0, \mathbf{u}^t), \\ \hat{\mathbf{v}}^t &= (0, 0, 0, \mathbf{v}^t), \\ \hat{Q}_{\mathbf{u},\mathbf{v}}^t &= (0, 0, 0, Q_{\mathbf{u},\mathbf{v}}^t). \end{aligned}$$

This gives a new fermionic expression for the NS-sector character.

4.6. Ramond sector characters. Let us set $\rho_1 = -zq^x$, $\rho_2 = -z^{-1}q^{x+1}$ in (4.9), which implies $a = q^{2x}$ and $s = 1$. Setting $x = 0$ and changing $j \rightarrow -j$ we obtain

$$(4.16) \quad \begin{aligned} &\sum_{n=0}^{\infty} \frac{(-z)_n (-z^{-1}q)_n}{(q)_{2n}} q^{-\mathcal{N}_{0,1}} F_{0,1}^{(p,p')}(2n, 1; q) \\ &= \frac{(-z)_{\infty} (-z^{-1}q)_{\infty}}{(q)_{\infty}^2} \sum_{j=-\infty}^{\infty} q^{jp(jp'+1)} \frac{1 - q^{2jp'+1}}{(1 + zq^{jp'})(1 + z^{-1}q^{jp'+1})}. \end{aligned}$$

Comparing with (4.7) we get

$$(4.17) \quad \hat{\chi}_{p,p'}^R(q, z) = z^{-\frac{c}{6}} q^{-\mathcal{N}_{0,1}} \sum_{n=0}^{\infty} \frac{(-z)_n (-z^{-1}q)_n}{(q)_{2n}} F_{0,1}^{(p,p')}(2n, 1; q).$$

Again using (2.12) in a similar way to the NS-sector and setting $z = 1$ we find

$$(4.18) \quad \hat{\chi}_{p,p'}^R(q) = 2q^{-\mathcal{N}_{0,1}+k_{1,1}} \sum_{n=0}^{\infty} \left(\frac{(-q)_{n-1}(-q)_n}{(q)_{2n}} \sum_{\mathbf{m} \equiv Q_{\mathbf{u},\mathbf{v}}} q^{\frac{1}{4}\mathbf{m}^t B \mathbf{m} - \frac{1}{2}A_{\mathbf{u},\mathbf{v}} \mathbf{m}} \times \prod_{j=1}^{t_{n_0+1}} \begin{bmatrix} n_j + m_j \\ m_j \end{bmatrix}_q \right).$$

Using

$$(x)_n = \sum_{k=0}^n (-x)^{(n-k)} q^{\frac{1}{2}(n-k)(n-k-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

and setting $m_0 = 2n$, equation (4.18) can be rewritten as

$$(4.19) \quad \hat{\chi}_{p,p'}^R(q) = 2q^{-\mathcal{N}_{0,1}+k_{1,1}} \sum_{\substack{m_0=0 \\ m_0 \text{ even}}}^{\infty} \sum_{k_1=0}^{\frac{m_0}{2}-1} \sum_{k_2=0}^{\frac{m_0}{2}} \sum_{\mathbf{m} \equiv Q_{\mathbf{u},\mathbf{v}}} q^{\frac{1}{4}(m_0^2+2k_1^2+2k_2^2-2m_0k_1-2m_0k_2)} \times q^{\frac{1}{4}\mathbf{m}^t B \mathbf{m} - \frac{1}{2}A_{\mathbf{u},\mathbf{v}} \mathbf{m} + \frac{1}{2}(k_1-k_2)} \frac{1}{(q)_{m_0}} \begin{bmatrix} \frac{m_0}{2} - 1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} \frac{m_0}{2} \\ k_2 \end{bmatrix}_q \prod_{j=1}^{t_{n_0+1}} \begin{bmatrix} n_j + m_j \\ m_j \end{bmatrix}_q.$$

Setting $\mathbf{p} = (k_1, k_2, m_0, \mathbf{m}) \in \mathbb{Z}^{t_{n_0+1}+3}$ this becomes

$$(4.20) \quad \hat{\chi}_{p,p'}^R(q) = 2q^{-\mathcal{N}_{0,1}+k_{1,1}} \sum_{\substack{\mathbf{p} \in \mathbb{Z}^{t_{n_0+1}+3} \\ p_i \equiv (\hat{Q}_{\mathbf{u},\mathbf{v}})_i, i \geq 3}} q^{\frac{1}{4}\mathbf{p}^t D \mathbf{p} - \frac{1}{2}\hat{A}_{\mathbf{u},\mathbf{v}} \mathbf{p}} \times \frac{1}{(q)_{p_3}} \prod_{j=1, j \neq 3}^{t_{n_0+1}+3} \begin{bmatrix} \frac{1}{2}(\mathcal{I}_D \mathbf{p} + \hat{\mathbf{u}} + \hat{\mathbf{v}})_j \\ p_j \end{bmatrix}_q,$$

with the same notations as in (4.15) except

$$\hat{A}_{\mathbf{u},\mathbf{v}} = (1, -1, 0, A_{\mathbf{u},\mathbf{v}}), \quad \hat{\mathbf{u}}^t = (-1, 0, 0, \mathbf{u}^t), \quad \hat{\mathbf{v}}^t = (-1, 0, 0, \mathbf{v}^t).$$

This gives a new fermionic expression of the new R-sector character.

5. CONCLUSION

In this paper we only considered the vacuum character for the $N = 2$ superconformal algebra with central charge $c = 3(1 - \frac{2p}{p'})$ with $p < p'$ in the NS-sector and the Ramond sector character derived from the vacuum character. We believe that similar Bailey flows exist for the general $N = 2$ superconformal characters, but explicit formulas are not yet available in the literature.

The astute reader might have noticed that unlike in section 3 we did not carry out the Bailey flow for the dual Bailey pair in section 4, the reason being that the fermionic formula

$F_{r,s}^{(p,p')}(L, b; q)$ for $p < p' < 2p$ and $r = b = 1$ are not given in [6, 7]. A formula however does appear in [26]. The matrix D in this case is

$$D = \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & B \end{array} \right).$$

Details will be available in [8].

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